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3D gauged supergravity from SU(2) reduction of $N = 1$ 6D supergravity

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Abstract: We obtain Yang-Mills $SU(2)\times G$ gauged supergravity in three dimensions from $SU(2)$ group manifold reduction of (1,0) six dimensional supergravity coupled to an anti-symmetric tensor multiplet and gauge vector multiplets in the adjoint of G. The reduced theory is consistently truncated to $N = 4$ 3D supergravity coupled to $4(1 + \dim G)$ bosonic and $4(1 + \dim G)$ fermionic propagating degrees of freedom. This is in contrast to the reduction in which there are also massive vector fields. The scalar manifold is $\mathbf{R} \times \frac{SO(3,\dim G)}{SO(3)\times SO(\dim G)}$, and there is a $SU(2) \times G$ gauge group. We then construct $N = 4$ Chern-Simons $(SO(3) \ltimes \mathbf{R}^3) \times (G \ltimes \mathbf{R}^{\dim G})$ three dimensional gauged supergravity with scalar manifold $\frac{SO(4,1+\dim G)}{SO(4)\times SO(1+\dim G)}$ and explicitly show that this theory is on-shell equivalent to the Yang-Mills $SO(3) \times G$ gauged supergravity theory obtained from the $SU(2)$ reduction, after integrating out the scalars and gauge fields corresponding to the translational symmetries $\mathbf{R}^3 \times \mathbf{R}^{\dim G}$.

Keywords: Supersymmetric Effective Theories, Supergravity Models.

1. Introduction

Three dimensional Chern-Simons gauged supergravities have a very rich structure and admit various types of gauging including non-semisimple and complex gauge groups [1, 2, 3, 4]. It has been shown in [4, 5] that non-semisimple Chern-Simons gaugings with gauge group $G \ltimes \mathbf{R}^{\dim G}$ are on-shell equivalent to semisimple Yang-Mills gaugings with gauge group G, including Chern-Simons couplings. This result makes it possible to obtain some of the Chern-Simons gauged supergravities with non-semisimple gauge groups from dimensional reductions of higher dimensional theories. For example, in [6, 7] the dimensional reduction of pure (1,0) six-dimensional supergravity on an $SU(2)$ group manifold [8] has been shown to give rise to a three dimensional gauged supergravity with $SU(2)$ Yang-Mills gauge group plus $SU(2)$ massive Chern-Simons vector fields and scalars in $GL(3,\mathbf{R})/SO(3)$, whose action has the structure essentially found in [5]. Also, the $N = 8$ theories studied in [9] and [10, 11], with $SO(4)$ and $SO(4)^2$ Yang-Mills gaugings respectively, are expected to arise from six dimensional (2,0) theory on S^3 and from IIB theory on $S^3 \times S^3 \times S^1$, respectively. Notice that in these examples, on the 3D supergravity side one can allow different gauge couplings for the two $SU(2)$'s in $SO(4)$, whereas from the S^3 dimensional reduction there is a single gauge coupling for the gauge fields arising from the isometries of S^3.

In this paper, we will consider an $N = 4$, 3D gauged supergravity where the scalar manifold is a single quaternionic space $\frac{SO(4,1+\dim G)}{SO(4)\times SO(1+\dim G)}$ and $(SO(3)\ltimes\mathbf{R}^3)\times(G\ltimes\mathbf{R}^{\dim G})$ Chern-Simons gauging, where G is an arbitrary semisimple group. We will show that this theory can be obtained from an $SU(2)$ reduction of a (1,0) six-dimensional supergravity coupled to a tensor multiplet and Yang-Mills multiplets of the gauge group G.

The six dimensional (1,0) gauged supergravity has been constructed in [12] and extended to couple to n_T anti-symmetric tensor multiplets, n_V vector multiplets and n_H hypermultiplets in [13]. The theory has been completed with quartic fermion terms in [14]. For earlier constructions of six dimensional (1,0) supergravity, we refer the reader to [15, 16]. We only consider the truncation of this extended theory to the ungauged theory, with $n_T = 1$ and $n_H = 0$, coupled to G Yang-Mills gauge fields. After reducing to three dimensions, we will show that the resulting theory is equivalent to the Chern-Simons gauged theory by reversing the procedure of [5].

The paper is organized as follows. In section 2, we review (1,0) six dimensional supergravity in order to set up our notations. In section 3, we will perform the $SU(2)$ group manifold reduction of (1,0) six dimensional supergravity coupled to an anti-symmetric tensor and G Yang-Mills multiplets and obtain $SU(2)\times G$ Yang-Mills gauged supergravity in three dimensions. The resulting theory contains $4(1+\dim G)$ bosons and $4(1+\dim G)$ fermions with the scalar manifold being $\mathbf{R}\times\frac{SO(3,\dim G)}{SO(3)\times SO(\dim G)}$. While it

is known that the most general $SU(2)$ reduction, including massive vector fields, is consistent, the novel feature of our work is that we make a further truncation by removing the massive vector fields and show that it is consistent. In section 4, we construct an $N=4$ Chern-Simons $(SO(3)\ltimes \mathbf{R}^3)\times(G\times \mathbf{R}^{\mathrm{dim}G})$ gauged theory with a scalar manifold $\frac{SO(4,1+\mathrm{dim}G)}{SO(4)\times SO(1+\mathrm{dim}G)}$ and show that it is indeed equivalent to $SO(3)\times G$ Yang-Mills gauged theory with scalar manifold $\mathbf{R}\times\frac{SO(3,\mathrm{dim}G)}{SO(3)\times SO(\mathrm{dim}G)}$ after removing $3+\mathrm{dim}G$ scalars corresponding to the translational symmetries. We finally give some conclusions and comments in section 5.

2. $N=(1,0)$ six dimensional supergravity

In this section, we briefly review and set up our notations for (1,0) six dimensional supergravity coupled to an antisymmetric tensor and G Yang-Mills multiplets. Pure (1,0) supergravity does not have a covariant action because of the self-duality of the three form field strength. Due to the anti-self duality of the 3-form field strength in the tensor multiplet, the coupled theory does admit a Lagrangian formulation. Six dimensional supergravity coupled to n_T anti-self dual tensor multiplets, n_V Yang-Mills vector multiplets and n_H hypermultiplets has been constructed in [13]. We are interested in the case of $n_T=1$ which admits a supersymmetric action. The theory considered here contains $N=1$ supergravity multiplet, one antisymmetric tensor multiplet and $\dim G$ Yang-Mills multiplets of an arbitrary gauge group G. We also assume that the group G commutes with the $SU(2)\sim Sp(1)$ R-symmetry group. The field content in this case is given by the graviton $e_M^{\hat{M}}$, gravitino ψ_M^A, third rank anti-symmetric tensor G_{3MNP}, scalar θ, spin $\frac{1}{2}$ fermion χ, G gauge fields A_M^I with $I=1,2,\ldots,\dim G$ and the G gauginos λ^I. The six dimensional spacetime indices are $M,N=0,\ldots 5$ with the tangent space indices $\hat{M},\hat{N}=0,\ldots 5$ while $A,B=1,2$ are $Sp(1)$ R-symmetry indices. The Lagrangian for this theory, with $\tilde{v}^z=0$, is given by [13]

$$\begin{aligned}
e^{-1}\mathcal{L} &= \frac{1}{4}R-\frac{1}{12}e^{2\theta}G_{3MNP}G_3{}^{MNP}-\frac{1}{4}\partial_M\theta\partial^M\theta-\frac{1}{2}\bar{\psi}_M\Gamma^{MNP}D_N\psi_P\\
&-\frac{1}{2}\bar{\chi}\Gamma^M D_M\chi-\frac{1}{4}v^z e^\theta F^I_{MN}F^{IMN}-v^z e^\theta\bar{\lambda}^I\Gamma^M D_M\lambda^I\\
&+\frac{1}{2}v^z e^\theta\bar{\chi}\Gamma^{MN}\lambda^I F^I_{MN}+\frac{1}{2}\bar{\psi}_M\Gamma^N\Gamma^M\chi\partial_N\theta-\frac{1}{2}v^z e^\theta\bar{\psi}_M\Gamma^{NP}\Gamma^M\lambda^I F^I_{NP}\\
&-\frac{1}{24}e^\theta G_{3MNP}\big[\bar{\psi}^L\Gamma_{[L}\Gamma^{MNP}\Gamma_{Q]}\psi^Q-2\bar{\psi}_L\Gamma^{MNP}\Gamma^L\chi-\bar{\chi}\Gamma^{MNP}\chi\\
&+2v^z e^\theta\bar{\lambda}^I\Gamma^{MNP}\lambda^I\big]
\end{aligned}\tag{2.1}$$

where $e=\sqrt{-g}$. We use the same metric signature as in [13], $(-+++++)$. The

supersymmetry transformations for various fields are [13]

$$
\begin{aligned}
\delta e_M^{\hat{M}} &= \bar{\epsilon}\Gamma^{\hat{M}}\psi_M, \\
\delta\psi_M &= D_M\epsilon + \frac{1}{24}e^{\theta}\Gamma^{NPQ}\Gamma_M G_{3NPQ}\epsilon - \frac{1}{16}\Gamma_M\chi\bar{\epsilon}\chi - \frac{3}{16}\Gamma^N\chi\bar{\epsilon}\Gamma_{MN}\chi \\
&\quad + \frac{1}{32}\Gamma_{MNP}\chi\bar{\epsilon}\Gamma^{NP}\chi - \frac{1}{16}v^z e^{\theta}(18\lambda^I\bar{\epsilon}\Gamma_M\lambda^I - 2\Gamma_{MN}\lambda^I\bar{\epsilon}\Gamma^N\lambda^I \\
&\quad + \Gamma_{NP}\lambda\bar{\epsilon}\Gamma_M^{\ NP}\lambda), \\
\delta b_{MN} &= 2v^z A^I_{[M}\delta A^I_{N]} - e^{-\theta}\bar{\epsilon}\Gamma_{[M}\psi_{N]} + \frac{1}{2}e^{-\theta}\bar{\epsilon}\Gamma_{MN}\chi, \\
\delta\theta &= \bar{\epsilon}\chi, \\
\delta\chi &= \frac{1}{2}\Gamma^M\partial_M\theta\epsilon - \frac{1}{12}e^{\theta}\Gamma^{MNP}G_{3MNP}\epsilon + \frac{1}{2}v^z e^{\theta}\Gamma^M\lambda^I(\bar{\epsilon}\Gamma_M\lambda^I), \\
\delta A^I_M &= -\bar{\epsilon}\Gamma_M\lambda^I, \\
\delta\lambda^I_A &= \frac{1}{4}\Gamma^{MN}F^I_{MN}\epsilon_A - C_z^{-1}v^z e^{\theta}\bar{\chi}_{(A}\lambda_{B)}\epsilon^B.
\end{aligned}
\tag{2.2}
$$

Notice that by our assumption on the gauge group G on the r.h.s. of $\delta\lambda^I_A$, the term $C^{AB}\epsilon_B$ is missing. One can see, using the formalism developed in the next section, that the presence of this term would imply a reduction of supersymmetry (if any) of the three dimensional theory. The bosonic field equations are given by [13]

$$
\begin{aligned}
& R_{MN} - \frac{1}{2}g_{MN}R - \frac{1}{3}e^{2\theta}\big(3G_{3MPQ}G_{3N}^{\ \ PQ} - \frac{1}{2}g_{MN}G_{3PQR}G_3^{\ PQR}\big) \\
& -\partial_M\theta\partial^M\theta + \frac{1}{2}g_{MN}\partial_P\theta\partial^P\theta - e^{\theta}\big(2F^{IP}_M F^I_{NP} - \frac{1}{2}g_{MN}F^I_{PQ}F^{IPQ}\big) = 0,
\end{aligned}
\tag{2.3}
$$

$$
e^{-1}\partial_M(eg^{MN}\partial_N\theta) - \frac{1}{2}e^{\theta}F^I_{MN}F^{IMN} - \frac{1}{3}e^{2\theta}G_{3MNP}G_3^{\ MNP} = 0, \tag{2.4}
$$

$$
D_N(ee^{\theta}F^{IMN}) + ee^{2\theta}G^{MNP}F^I_{NP} = 0, \tag{2.5}
$$

$$
D_M(ee^{2\theta}G_3^{\ MNP}) = 0. \tag{2.6}
$$

We also choose $v^z = 1$ from now on. The three form field strength is

$$
G_3 = db + F^I \wedge A^I - \frac{1}{6}g_2 f_{IJK}A^I \wedge A^J \wedge A^K \tag{2.7}
$$

where g_2 and f_{IJK} are coupling and structure constants of the gauge group G, respectively. The equations of motion for various fermions can be found in [13]. We will not repeat them here because they will not be needed in this work. In the next section, we will give the reduction ansatz and perform the dimensional reduction of this theory on the $SU(2)$ group manifold.

3. 3D $SU(2)\times G$ gauged supergravity from (1,0) six dimensional supergravity on S^3

In this section, we study Kaluza-Klein reduction of the (1,0) six dimensional supergravity coupled to an anti-symmetric and Yang-Mills vector multiplets with a gauge group G on $SU(2)$ group manifold. The result is the $N=4$, $SU(2)\times G$ gauged supergravity in three dimensions. The (1,0) six dimensional supergravity has been obtained from various compactifications of string and M theory e.g. [17], [18]. These compactifications have been used to study many aspects of string dualities in six dimensions e.g. [19], [20].

3.1 Reduction ansatz on $SU(2)$ group manifold

We now give our reduction ansatz. We will put a hat on all the six dimensional fields from now on. We use the following reduction ansatz:

$$\begin{aligned}
d\hat{s}^2 &= e^{2f}ds^2 + e^{2g}h_{\alpha\beta}\nu^\alpha\nu^\beta\,,\\
\hat{A}^I &= A^I + A^I_\alpha\nu^\alpha, \qquad \nu^\alpha = \sigma^\alpha - g_1 A^\alpha\,,\\
\hat{F}^I &= d\hat{A}^I + \frac{1}{2}g_2 f_{IJK}\hat{A}^J\wedge\hat{A}^K\\
&= F^I - g_1 A^I_\alpha F^\alpha + \mathcal{D}A^I_\alpha\wedge\nu^\alpha + \frac{1}{2}(g_2 A^J_\alpha A^K_\beta f_{IJK} - \epsilon_{\alpha\beta\gamma}A^I_\gamma)\nu^\alpha\wedge\nu^\beta
\end{aligned} \tag{3.1}$$

where the $SU(2)\times G$ covariant derivative is given by

$$\mathcal{D}A^I_\alpha = dA^I_\alpha + g_1\epsilon_{\alpha\beta\gamma}A^\beta A^I_\gamma + g_2 f_{IJK}A^J A^K_\alpha. \tag{3.2}$$

The three dimensional field strength $F^I = dA^I + \frac{1}{2}g_2 f_{IJK}A^J\wedge A^K$. From the metric, we can read off the vielbein components

$$\hat{e}^a = e^f e^a,\ \ \hat{e}^i = e^g L^i_\alpha\nu^\alpha \text{ with } h_{\alpha\beta} = L^i_\alpha L^i_\beta. \tag{3.3}$$

The left-invariant $SU(2)$ 1-forms σ^α satisfy

$$d\sigma^\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\gamma}\sigma^\beta\wedge\sigma^\gamma\,. \tag{3.4}$$

The $\epsilon_{\alpha\beta\gamma}$ and f_{IJK} are the $SU(2)$ and G structure constants, respectively. The metric $h_{\alpha\beta}$ and a (3×3) matrix L^i_α are unimodular. The spin connections are given by [6]

$$\begin{aligned}
\hat{\omega}_{ab} &= \omega_{ab} + e^{-f}(\partial_b f\eta_{ac} - \partial_a f\eta_{bc})\hat{e}^c + \frac{1}{2}g_1 e^{g-2f}F^i_{ab}\hat{e}^i,\\
\hat{\omega}_{ai} &= -e^{-f}P_{aij}\hat{e}^j - e^{-f}\partial_a g\hat{e}^i + e^{g-2f}F^i_{ab}\hat{e}^b,\\
\hat{\omega}_{ij} &= e^{-f}Q_{aij}\hat{e}^a + \frac{1}{2}e^{-g}(T^{kl}\epsilon_{ijl} + T^{jl}\epsilon_{ikl} - T^{il}\epsilon_{jkl})\hat{e}^k
\end{aligned} \tag{3.5}$$

where

$$
\begin{aligned}
P_{aij} &= \frac{1}{2}\left[(L^{-1})_i^\alpha D_a L_\alpha^j + (L^{-1})_j^\alpha D_a L_\alpha^i\right] = \frac{1}{2}(L^{-1})_i^\alpha (L^{-1})_j^\beta D_a h_{\alpha\beta}, \\
Q_{aij} &= \frac{1}{2}\left[(L^{-1})_i^\alpha D_a L_\alpha^j - (L^{-1})_j^\alpha D_a L_\alpha^i\right], \\
F^i &= L_\alpha^i F^\alpha, \qquad T^{ij} = L_\alpha^i L_\alpha^j, \qquad DL_\alpha^i = dL_\alpha^i - g_1 \epsilon_{\alpha\beta\gamma} A^\gamma L_\beta^i.
\end{aligned} \tag{3.6}
$$

We use the same conventions as in [6] namely

$$
\begin{aligned}
F^\alpha &= dA^\alpha + \frac{1}{2} g_1 \epsilon_{\alpha\beta\gamma} A^\beta \wedge A^\gamma, \\
DF^\alpha &= dF^\alpha + g_1 \epsilon_{\alpha\beta\gamma} A^\beta \wedge A^\gamma = 0, \\
D\nu^\alpha &= d\nu^\alpha + g_1 \epsilon_{\alpha\beta\gamma} A^\beta \wedge \nu^\gamma = -g_1 F^\alpha - \frac{1}{2}\epsilon_{\alpha\beta\gamma} \nu^\beta \wedge \nu^\gamma.
\end{aligned} \tag{3.7}
$$

The indices $(M, \hat{M})$ reduce to (μ, a) in three dimensions while the S^3 part is described by indices (α, i). The ansatz for $\hat{G}_3$ is

$$
\begin{aligned}
\hat{G}_3 = {} & h\varepsilon_3 + a\epsilon_{\alpha\beta\gamma}\nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma + \epsilon_{\alpha\beta\gamma} C^\alpha \wedge \nu^\beta \wedge \nu^\gamma + H^\alpha \wedge \nu^\alpha \\
& + \hat{F}^I \wedge \hat{A}^I - \frac{1}{6} g_2 f_{IJK} \hat{A}^I \wedge \hat{A}^J \wedge \hat{A}^K.
\end{aligned} \tag{3.8}
$$

The first line in (3.8) is the $d\hat{b}$ which must be closed. This requires that

$$
H^\alpha = 2DB^\alpha - 6ag_1 F^\alpha. \tag{3.9}
$$

We also choose the one form $C_\alpha = \frac{1}{2} A_\alpha^I A^I$ to further simplify the ansatz and truncate the vector field C^α out. Putting all together, we end up with the following $\hat{G}_3$ ansatz

$$
\begin{aligned}
\hat{G}_3 = {} & \tilde{h}\varepsilon_3 + \bar{F}^\alpha \wedge \nu^\alpha + \frac{1}{2} K_{\alpha\beta} \wedge \nu^\alpha \wedge \nu^\beta \\
& + \frac{1}{6}\tilde{a}\epsilon_{\alpha\beta\gamma}\nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma
\end{aligned} \tag{3.10}
$$

where $\tilde{h} = h\varepsilon_3 + \tilde{F}^I \wedge A^I - \frac{1}{6} g_2 A^I \wedge A^J \wedge A^K f_{IJK}$. We have defined the following quantities

$$
\begin{aligned}
\bar{F}^\alpha &= A_\alpha^I (\tilde{F}^I + F^I) - 6ag_1 F^\alpha, \qquad \tilde{F}^I = F^I - g_1 A_\alpha^I F^\alpha, \\
K_{\alpha\beta} &= A_\beta^I \mathcal{D} A_\alpha^I - A_\alpha^I \mathcal{D} A_\beta^I, \\
\tilde{a} &= 6a - A_\alpha^I A_\alpha^I + \frac{1}{3} g_2 A^3, \qquad A^3 \equiv A_\alpha^I A_\beta^J A_\gamma^K f_{IJK} \epsilon_{\alpha\beta\gamma},
\end{aligned} \tag{3.11}
$$

and a is a constant. The ansatz for the Yang-Mills fields can be rewritten as

$$
\hat{F}^I = \tilde{F}^I + \mathcal{D} A_\alpha^I \wedge \nu^\alpha + \frac{1}{2}\mathcal{F}_{\alpha\beta}^I \nu^\alpha \wedge \nu^\beta \tag{3.12}
$$

where $\mathcal{F}^I_{\alpha\beta} = g_2 A^J_\alpha A^K_\beta f_{IJK} - A^I_\gamma \epsilon_{\alpha\beta\gamma}$.

The volume form in three dimensions is defined by

$$\varepsilon_3 = \frac{1}{6} e^{3f} \epsilon_{abc} e^a \wedge e^b \wedge e^c \equiv e^{3f} \omega_3 \,. \tag{3.13}$$

The six dimensional gamma matrices decompose as [6]

$$\begin{aligned} \Gamma^{\hat{A}} &= (\Gamma^a, \Gamma^i), \qquad \Gamma^a = \gamma^a \otimes \mathbb{I}_2 \otimes \sigma_1, \\ \Gamma^i &= \mathbb{I}_2 \otimes \gamma^i \otimes \sigma_2, \qquad \Gamma_7 = \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma_3 \\ \gamma^{abc} &= \epsilon^{abc}, \qquad \gamma^{ijk} = i\epsilon^{ijk}, \qquad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \;\; \{\gamma_i, \gamma_j\} = 2\delta_{ij}. \end{aligned} \tag{3.14}$$

The conventions are $\eta_{AB} = (-+++++)$, $\eta_{ab} = (-++)$ and $\epsilon^{012} = \epsilon^{345} = 1$. We further choose

$$\gamma^0 = i\tilde{\sigma}^2, \; \gamma^1 = \tilde{\sigma}^1, \; \gamma^2 = \tilde{\sigma}^3, \qquad \gamma^i = \tau^i \tag{3.15}$$

where $\tilde{\sigma}^i$, τ^i, $i = 1,2,3$ are the usual Pauli matrices. Also the chirality condition $\Gamma_7 \epsilon^A = \epsilon^A$ becomes $\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma_3 \epsilon^A = \epsilon^A$.

Before proceeding further, let us count the number of degrees of freedom. Table 3.1 shows all three dimensional fields arising from the six dimensional ones. From table

6D fields	3D fields	3D number of degrees of freedom
$\hat{g}_{MN}$	$g_{\mu\nu}$	non propagating
	A^α_μ	3
	$h_{\alpha\beta}$	5
	g	1
$\hat{b}_{MN}$	$b_{\mu\nu}$	non propagating
	$b_{\mu\alpha}$	3
	$b_{\alpha\beta}$	3
$\hat{\theta}$	θ	1
$\hat{A}^I_M$	A^I_μ	dim$G$
	A^I_α	$3\,\mathrm{dim}G$
$\hat{\psi}_M$	ψ_μ	non propagating
	ψ_i	12
$\hat{\lambda}^I$	λ^I	$4\,\mathrm{dim}G$
$\hat{\chi}$	χ	4

Table 1: Three dimensional fields and the associated number of degrees of freedom.

3.1, there are $16 + 4\mathrm{dim}G$ bosonic and $16 + 4\mathrm{dim}G$ fermionic degrees of freedom in the full reduced theory. In this counting, each six dimensional fermion gives rise to 4 three

dimensional fermions. In the reduction of the six dimensional theory, the component $\hat{b}_{\mu\alpha}$ will give rise to massive vector fields in three dimensions. Our goal is to truncate this theory to obtain a three dimensional $N = 4$ gauged supergravity involving only gravity, scalars and gauge fields without massive vector fields. The resulting theory will have $4(1 + \dim G)$ bosonic and $4(1 + \dim G)$ fermionic propagating degrees of freedom. To achieve this, we need to truncate 12 degrees of freedom out. From the $\hat{G}_3$ ansatz expressed entirely in terms of gauge fields, scalars coming from the gauge fields in six dimensions and constants, we see that all the fields coming from $\hat{b}_{MN}$ have been truncated out. This accounts for 6 degrees of freedom. We will see below that $h_{\alpha\beta}$ and θ, comprising 6 degrees of freedom, will be truncated, too.

In the fermionic sector, we find that the truncation is given by

$$\hat{\psi}_i - \frac{1}{2}\Gamma_i\hat{\chi} - 2e^{\theta-g}A^I_\alpha(L^{-1})_i^\alpha\hat{\lambda}^I = 0. \tag{3.16}$$

Indeed, this removes 12 fermionic degrees of freedom. We have checked that this truncation is compatible with supersymmetry to leading order in fermions. We refer the readers to appendix A for the detail of this computation. From appendix A, we find that

$$\delta\hat{\psi}_i - \frac{1}{2}\Gamma_i\delta\hat{\chi} - 2e^{\theta-g}A^I_\alpha(L^{-1})_i^\alpha\delta\hat{\lambda}^I = 0 \tag{3.17}$$

provided that

$$h_{\alpha\beta} = e^{\theta-2g}(12a\delta_{\alpha\beta} - 2A^I_\alpha A^I_\beta) \equiv e^{\theta-2g}N_{\alpha\beta}. \tag{3.18}$$

This is the truncation in the bosonic sector. From (3.18), it follows that

$$\theta = 2g - \frac{1}{3}\ln N \tag{3.19}$$

where $N \equiv \det(N_{\alpha\beta})$. Also, from (3.18), it can be easily checked that

$$\delta[h_{\alpha\beta} - e^{\theta-2g}(12a\delta_{\alpha\beta} - 2A^I_\alpha A^I_\beta)] = 0 \tag{3.20}$$

to leading order in fermions by using (3.16). The detail can be found in appendix A. So, the relation (3.18) is compatible with supersymmetry. Equations (3.18) and (3.19) give another truncation in the bosonic sector and remove 6 degrees of freedom. The bosonic degree of freedoms are then given by $1 + 3\dim(\mathrm{G})$ scalars, g and A^I_α, and $3 + \dim(\mathrm{G})$ vectors, A^α and A^I. So, the reduced theory contains $4(1 + \dim G)$ propagating degrees of freedom and involves only gravity, scalars and vector gauge fields.

We now check the consistency of the six dimensional field equations. It is convenient to rewrite equations (2.4), (2.5) and (2.6) in differential forms. We find that these

equations can be written as

$$\hat{\mathcal{D}}(e^{2\hat{\theta}}\hat{*}\hat{G}_3) = 0, \tag{3.21}$$

$$\hat{\mathcal{D}}(e^{\hat{\theta}}\hat{*}\hat{F}^I) - 2e^{2\hat{\theta}}\hat{*}\hat{G}_3 \wedge \hat{F}^I = 0, \tag{3.22}$$

$$\hat{d}\hat{*}\hat{d}\hat{\theta} + e^{\hat{\theta}}\hat{*}\hat{F}^I \wedge \hat{F}^I + 2e^{2\hat{\theta}}\hat{*}\hat{G}_3 \wedge \hat{G}_3 = 0\,. \tag{3.23}$$

In order to obtain the canonical Einstein-Hilbert term in three dimensions, we choose $f = -3g$ from now on. Before giving equations of motion, we give here the Hodge dual of $\hat{F}^I$ and $\hat{G}_3$

$$\begin{aligned}\hat{*}\hat{F}^I &= \frac{1}{3!}e^{6g} * \tilde{F}^I \epsilon_{\alpha\beta\gamma}\nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma + \frac{1}{2}e^{-2g}h^{\alpha\delta}\epsilon_{\beta\gamma\delta} * \mathcal{D}A^I_\alpha \wedge \nu^\beta \wedge \nu^\gamma \\ &\quad + \frac{1}{2}e^{-10g}\mathcal{F}^I_{\alpha\beta}h_{\gamma\delta}\epsilon_{\alpha\beta\delta}\omega_3 \wedge \nu^\gamma, \end{aligned} \tag{3.24}$$

$$\begin{aligned}\hat{*}\hat{G}_3 &= -\frac{1}{3!}e^{3g}\tilde{h}\epsilon_{\alpha\beta\gamma}\nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma + \frac{1}{2}e^{4g}h^{\alpha\delta}\epsilon_{\beta\gamma\delta} * \bar{F}^\alpha \wedge \nu^\beta \wedge \nu^\gamma \\ &\quad - \frac{1}{2}e^{-4g}\epsilon_{\alpha\beta\gamma}h_{\gamma\delta} * K_{\alpha\beta} \wedge \nu^\delta + \tilde{a}e^{-12g}\omega_3\ . \end{aligned} \tag{3.25}$$

The $\hat{*}$ and $*$ are Hodge dualities in six and three dimensions, respectively. After using our ansatz in (3.21), (3.22) and (3.23), we find the following set of equations

$$\mathcal{D}(e^{2\theta+3g}\tilde{h}) = 0\,, \tag{3.26}$$

$$\mathcal{D}(e^{\theta+6g}N^{\alpha\beta} * \bar{F}^\beta) + g_1c_1F^\alpha + \frac{1}{2}\epsilon_{\alpha\beta\gamma}N^{\alpha'\beta}N^{\beta'\gamma} * K_{\alpha'\beta'} = 0\,, \tag{3.27}$$

$$\mathcal{D}(N^{\alpha\gamma}N^{\beta\delta} * K_{\alpha\beta}) - g_1e^{\theta+6g}(N^{\alpha\gamma} * \bar{F}^\alpha \wedge F^\delta - N^{\alpha\delta} * \bar{F}^\alpha \wedge F^\gamma) = 0\,, \tag{3.28}$$

$$\begin{aligned}&\mathcal{D}(e^{\theta+6g} * \tilde{F}^I) + 2c_1\tilde{F}^I - 2e^{\theta+6g}N^{\alpha\alpha'} * \bar{F}^{\alpha'} \wedge \mathcal{D}A^I_\alpha \\ &\quad + N^{\alpha\alpha'}N^{\beta\beta'}\mathcal{F}^I_{\alpha\beta} * K_{\alpha'\beta'} + g_2N^{\alpha\delta}f_{IJK}A^J_\delta * \mathcal{D}A^K_\alpha = 0\,, \end{aligned} \tag{3.29}$$

$$\begin{aligned}&\mathcal{D}(N^{\alpha\beta} * \mathcal{D}A^I_\beta) + g_1e^{\theta+6g} * \tilde{F}^I \wedge F^\alpha - 2e^{\theta+6g}N^{\alpha\beta} * \bar{F}^\beta \wedge \tilde{F}^I \\ &\quad + 2N^{\alpha\alpha'}N^{\beta\beta'} * K_{\alpha'\beta'} \wedge \mathcal{D}A^I_\beta + \frac{1}{2}e^{-\theta-6g}N^{\alpha'\beta}N^{\beta'\gamma}\mathcal{F}^I_{\alpha'\beta'}\epsilon_{\alpha\beta\gamma}\omega_3 \\ &\quad - \tilde{a}e^{2\theta-12g}\epsilon_{\alpha\beta\gamma}\mathcal{F}^I_{\beta\gamma}\omega_3 + g_2f_{IJK}e^{-\theta-6g}A^J_\beta\mathcal{F}^K_{\alpha'\beta'}N^{\alpha'\beta}N^{\alpha\beta'}\omega_3 = 0\,, \end{aligned} \tag{3.30}$$

$$\begin{aligned}&2d * dg - \frac{1}{3}d\ln N + e^{\theta+6g} * \tilde{F}^I \wedge \tilde{F}^I + N^{\alpha\alpha'} * \mathcal{D}A^I_{\alpha'} \wedge \mathcal{D}A^I_\alpha \\ &\quad + \frac{1}{2}e^{\theta+6g}N^{\alpha\alpha'} * \bar{F}^{\alpha'} \wedge \bar{F}^\alpha + \frac{1}{2}N^{\alpha\alpha'}N^{\beta\beta'} * K_{\alpha'\beta'} \wedge K_{\alpha\beta} + c_1^2e^{-2\theta-12g}\omega_3 \\ &\quad + \frac{1}{2}e^{-\theta-6g}N^{\alpha\alpha'}N^{\beta\beta'}\mathcal{F}^I_{\alpha\beta}\mathcal{F}^I_{\alpha'\beta'}\omega_3 + \tilde{a}^2e^{-12g}\omega_3 = 0\,, \end{aligned} \tag{3.31}$$

where we have used the summation convention on $\alpha, \beta, \ldots$ regardless their upper or lower positions, and $N^{\alpha\beta} \equiv (N^{-1})_{\alpha\beta}$. We have also used the solution for equation (3.26) namely

$$\tilde{h}e^{2\theta+3g} = c_1 \tag{3.32}$$

with a constant c_1 in other equations. Equation (3.28) can be obtained by multiplying (3.30) by $A^I_{\beta'}N^{\beta\beta'}$ and antisymmetrizing in α and β. By using the explicit forms of the Ricci tensors given in [6], the scalar equation (3.30), after multiplied by $e^{8g-\theta}N^{\beta\beta'}A^I_{\beta'}$ and symmetrized in α and β, is the same as component ij of the Einstein equation with the trace part of Einstein equation taking care of equation (3.31). Component ab the Einstein equation will give three dimensional Einstein equation which we will not give the explicit form here. Equation (3.27) gives Yang-Mills equations for A^α. The combination [(3.29) $+2A^I_\alpha$(3.27)] gives Yang-Mills equations for A^I

$$\begin{aligned}&\mathcal{D}[e^{\theta+6g}[(\delta_{IJ}+4A^I_\alpha A^J_\beta N^{\alpha\beta})F^J-24g_1aA^I_\alpha N^{\alpha\beta}F^\beta]]+2c_1F^I\\&+g_2f_{IJK}N^{\alpha\beta}A^J_\beta*\mathcal{D}A^K_\alpha+g_2f_{IJK}N^{\alpha\alpha'}N^{\beta\beta'}A^J_\alpha A^K_\beta*K_{\alpha'\beta'}=0\,.\end{aligned}\tag{3.33}$$

We have checked that the equation for F^α is the same as component ai of the Einstein equation. So, there are two Yang-Mills equations for F^α and F^I, one equation for g and one equation for A^I_α. All six dimensional field equations are satisfied by our ansatz.

3.2 Three dimensional gauged supergravity Lagrangian

All three dimensional equations of motion obtained in the previous subsection can be obtained from the following Lagrangian, with $\hat{e}=ee^{3f+3g}=ee^{-6g}$,

$$\begin{aligned}\mathcal{L}=&\frac{1}{4}R*\mathbf{1}-\frac{1}{2}N^{-\frac{1}{3}}e^{8g}\big[(\delta_{IJ}+4A^I_\alpha A^J_\beta N^{\alpha\beta})*F^I\wedge F^J-48ag_1N^{\alpha\beta}A^I_\beta*F^\alpha\wedge F^I\\&+6ag_1^2(24aN^{\alpha\beta}-\delta_{\alpha\beta})*F^\alpha\wedge F^\beta\big]-*d\big(2g-\frac{1}{12}\ln N\big)\wedge d\big(2g-\frac{1}{12}\ln N\big)\\&-\frac{1}{2}N^{\alpha\beta}*\mathcal{D}A^I_\alpha\wedge\mathcal{D}A^I_\beta-N^{\alpha\alpha'}N^{\beta\beta'}A^I_\beta A^J_{\beta'}*\mathcal{D}A^I_\alpha\wedge\mathcal{D}A^J_{\alpha'}-V*\mathbf{1}+\mathcal{L}_{\rm CS}\end{aligned}\tag{3.34}$$

which is the same as the dimensional reduction of the Lagrangian

$$\mathcal{L}_B=\frac{1}{4}\hat{R}\hat{*}\mathbf{1}-\frac{1}{4}\hat{*}\hat{d}\hat{\theta}\wedge\hat{d}\hat{\theta}-\frac{1}{2}e^{2\hat{\theta}}\hat{*}\hat{G}_3\wedge\hat{G}_3-\frac{1}{2}e^{\hat{\theta}}\hat{*}\hat{F}^I\wedge\hat{F}^I\tag{3.35}$$

together with the Chern-Simons terms. The scalar potential and the Chern-Simons Lagrangian are given by

$$\begin{aligned}V=&\frac{1}{4}\big[N^{-\frac{2}{3}}(N_{\alpha\beta}N_{\alpha\beta}-\frac{1}{2}N_{\alpha\alpha}N_{\beta\beta})+2N^{-\frac{2}{3}}e^{-8g}\tilde{a}^2\\&+N^{\frac{1}{3}}e^{-8g}N^{\alpha\alpha'}N^{\beta\beta'}\mathcal{F}^I_{\alpha\beta}\mathcal{F}^I_{\alpha'\beta'}-2c_1^2N^{\frac{2}{3}}e^{-16g}\big],\end{aligned}\tag{3.36}$$

$$\begin{aligned}\mathcal{L}_{\rm CS}=&2c_1\big(F^I\wedge A^I-\frac{1}{6}g_2f_{IJK}A^I\wedge A^J\wedge A^K\big)\\&-12ag_1^2c_1\big(F^\alpha\wedge A^\alpha-\frac{1}{6}g_1\epsilon_{\alpha\beta\gamma}A^\alpha\wedge A^\beta\wedge A^\gamma\big)\,.\end{aligned}\tag{3.37}$$

In order to make formulae simpler and the symmetries of the scalar manifold more transparent, we make the following rescalings. We first restore the coupling g_1 in the appropriate places by setting

$$a = \frac{\bar{a}}{g_1^2}. \tag{3.38}$$

We can then remove the constant a by setting

$$\begin{aligned}
c_1 &= \frac{\bar{c}_1}{6\bar{a}}, \qquad e^g = \frac{e^{\bar{g}}}{g_1^{\frac{1}{4}}}, \qquad A^I_\alpha = \frac{\sqrt{6\bar{a}}}{g_1}\bar{A}^I_\alpha, \\
g_2 &= \frac{\bar{g}_2}{\sqrt{6\bar{a}}}, \qquad N_{\alpha\beta} = \frac{6\bar{a}}{g_1^2}\bar{N}_{\alpha\beta}, \qquad e^\theta = \frac{g_1^{\frac{3}{2}}}{6\bar{a}}e^{\bar{\theta}} \\
&\text{and} \qquad A^I = \sqrt{6\bar{a}}\bar{A}^I\,.
\end{aligned} \tag{3.39}$$

After removing all the bars, we obtain the Lagrangian

$$\begin{aligned}
\mathcal{L} = \frac{1}{4}R * \mathbf{1} &- \frac{1}{2}e^{2\sqrt{2}\Phi}[(\delta_{IJ} + 4N^{\alpha\beta}A^I_\alpha A^J_\beta) * F^I \wedge F^J - 8N^{\alpha\beta}A^I_\beta * F^\alpha \wedge F^I \\
&+(4N^{\alpha\beta} - \delta_{\alpha\beta}) * F^\alpha \wedge F^\beta] - \frac{1}{2} * d\Phi \wedge d\Phi - \frac{1}{2}N^{\alpha\beta} * \mathcal{D}A^I_\alpha \wedge \mathcal{D}A^I_\beta \\
&-N^{\alpha\alpha'}N^{\beta\beta'}A^I_\beta A^J_{\beta'} * \mathcal{D}A^I_\alpha \wedge \mathcal{D}A^J_{\alpha'} - V + \mathcal{L}_{\text{CS}}
\end{aligned} \tag{3.40}$$

where we have introduced the canonically normalized scalar for the gauge singlet combination

$$\Phi = 2\sqrt{2}g - \frac{\sqrt{2}}{12}\ln N\,. \tag{3.41}$$

The scalar potential and Chern-Simons terms are now

$$\begin{aligned}
V &= \frac{1}{4}\big[g_1^2 N^{-1}e^{-2\sqrt{2}\Phi}(N_{\alpha\beta}N_{\alpha\beta} - \frac{1}{2}N_{\alpha\alpha}N_{\beta\beta}) + 2N^{-1}e^{-2\sqrt{2}\Phi}\tilde{a}^2 - 2c_1^2 e^{-4\sqrt{2}\Phi} \\
&\quad +e^{-2\sqrt{2}\Phi}N^{\alpha\alpha'}N^{\beta\beta'}\mathcal{F}^I_{\alpha\beta}\mathcal{F}^I_{\alpha'\beta'}\big],
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
\mathcal{L}_{\text{CS}} &= 2c_1\Big[F^I \wedge A^I - \frac{1}{6}g_2 f_{IJK}A^I \wedge A^J \wedge A^K \\
&\quad -\big(F^\alpha \wedge A^\alpha \wedge -\frac{1}{6}g_1\epsilon_{\alpha\beta\gamma}A^\alpha \wedge A^\beta \wedge A^\gamma\big)\Big]
\end{aligned} \tag{3.43}$$

with

$$\begin{aligned}
\tilde{a} &= g_1(1 - A^I_\alpha A^I_\alpha) + \frac{1}{3}g_2 A^3\,, \\
\mathcal{F}^I_{\alpha\beta} &= g_2 A^J A^K f_{IJK} - g_1\epsilon_{\alpha\beta\gamma}A^I_\gamma\,.
\end{aligned} \tag{3.44}$$

We first look at the scalar matrix appearing in the gauge kinetic terms

$$\mathbf{M} = \begin{pmatrix} \mathcal{M}_{\alpha\beta} & \mathcal{M}_{\alpha J} \\ \mathcal{M}_{I\beta} & \mathcal{M}_{IJ} \end{pmatrix} = e^{2\sqrt{2}\Phi} \begin{pmatrix} 4N^{\alpha\beta} - \delta_{\alpha\beta} & -4N^{\alpha\beta} A^J_\beta \\ -4N^{\alpha\beta} A^I_\alpha & \delta_{IJ} + 4N^{\alpha\beta} A^I_\alpha A^J_\beta \end{pmatrix}. \tag{3.45}$$

Introducing the matrix notation for ${A^I}_\alpha \equiv \mathbf{A}$ which is an $n \times 3$, $n = \dim G$, matrix and denoting $\mathbb{I}_3 = \mathbb{I}_{3\times 3}$ etc, we find

$$N_{\alpha\beta} \equiv \mathbf{N} = 2\Big(\mathbb{I}_3 - \mathbf{A}^t\mathbf{A}\Big), \qquad N^{\alpha\beta} \equiv \mathbf{N}^{-1} = \frac{1}{2\Big(\mathbb{I}_3 - \mathbf{A}^t\mathbf{A}\Big)}, \tag{3.46}$$

$$\mathbf{M} = e^{2\sqrt{2}\Phi} \begin{pmatrix} \frac{\mathbb{I}_3 + \mathbf{A}^t\mathbf{A}}{\mathbb{I}_3 - \mathbf{A}^t\mathbf{A}} & -2\frac{1}{\mathbb{I}_3 - \mathbf{A}^t\mathbf{A}}\mathbf{A}^t \\ -2\frac{1}{\mathbb{I}_n - \mathbf{A}\mathbf{A}^t}\mathbf{A} & \frac{\mathbb{I}_n + \mathbf{A}^t\mathbf{A}}{\mathbb{I}_n - \mathbf{A}^t\mathbf{A}} \end{pmatrix}. \tag{3.47}$$

It follows that

$$\mathbf{M}^{-1} = e^{-2\sqrt{2}\Phi} \begin{pmatrix} \frac{\mathbb{I}_3 + \mathbf{A}^t\mathbf{A}}{\mathbb{I}_3 - \mathbf{A}^t\mathbf{A}} & 2\frac{1}{\mathbb{I}_3 - \mathbf{A}^t\mathbf{A}}\mathbf{A}^t \\ 2\frac{1}{\mathbb{I}_n - \mathbf{A}\mathbf{A}^t}\mathbf{A} & \frac{\mathbb{I}_n + \mathbf{A}^t\mathbf{A}}{\mathbb{I}_n - \mathbf{A}^t\mathbf{A}} \end{pmatrix}. \tag{3.48}$$

The scalars form the coset space $\mathbf{R} \times \frac{SO(3,n)}{SO(3)\times SO(n)}$ with the factor $\mathbf{R}$ corresponding to Φ. The scalar kinetic terms give rise to the metric on $\mathbf{R} \times \frac{SO(3,n)}{SO(3)\times SO(n)}$

$$\begin{aligned} &-\frac{1}{2} * d\Phi \wedge d\Phi - \frac{1}{2} N^{\alpha\beta} * \mathcal{D}A^I_\alpha \wedge \mathcal{D}A^I_\beta - N^{\alpha\alpha'} N^{\beta\beta'} A^I_\beta A^J_{\beta'} * \mathcal{D}A^I_\alpha \wedge \mathcal{D}A^J_{\alpha'} \\ &= -\frac{1}{2} * d\Phi \wedge d\Phi - \frac{1}{4}\mathrm{Tr}\bigg(\frac{1}{\mathbb{I}_3 - \mathbf{A}^t\mathbf{A}} * \mathcal{D}\mathbf{A}^t \wedge \frac{1}{\mathbb{I}_n - \mathbf{A}\mathbf{A}^t}\mathcal{D}\mathbf{A}\bigg). \end{aligned} \tag{3.49}$$

With all these results, the Lagrangian can be simply written as

$$\begin{aligned} \mathcal{L} = &\frac{1}{4} R * \mathbf{1} - \frac{1}{2} * d\Phi \wedge d\Phi - \frac{1}{4}\mathrm{Tr}\bigg(\frac{1}{\mathbb{I}_3 - \mathbf{A}^t\mathbf{A}} * \mathcal{D}\mathbf{A}^t \wedge \frac{1}{\mathbb{I}_n - \mathbf{A}\mathbf{A}^t}\mathcal{D}\mathbf{A}\bigg) \\ &-\frac{1}{2} e^{2\sqrt{2}\Phi} \mathcal{M}_{\mathcal{A}\mathcal{B}} * F^{\mathcal{A}} \wedge F^{\mathcal{B}} - V + \mathcal{L}_{\mathrm{CS}} \end{aligned} \tag{3.50}$$

where $\mathcal{A}, \mathcal{B} = (\alpha, I)$.

We now come to supersymmetries of our truncated theory. We will show that this truncation is indeed compatible with supersymmetry namely supersymmetry transformations of various components of $\hat{b}_{MN}$ must be consistent with our specific choices of $C^\alpha = \frac{1}{2} A^I_\alpha A^I$. This ensures that all the truncated fields will not be generated via supersymmetry. For the filed $\hat{b}_{\mu\nu}$, we have eliminated it by using the equation of motion for $\hat{G}_3$ in (3.26). Because of its non propagating nature, we do not need to worry about it. For $\delta\hat{G}_{3\mu\alpha\beta}$ and $\delta\hat{G}_{3\mu\nu\alpha}$, we have checked that they are consistent. The detail of this

check can be found in appendix A. We have then verified that our truncated theory is a supersymmetric theory. We will also give a confirmation to this claim in the next section in which we will show that this theory is on-shell equivalent to a manifestly supersymmetric $(SO(3) \ltimes \mathbf{R}^3) \times (G \ltimes \mathbf{R}^n)$ Chern-Simons gauged supergravity.

The final issue we should add here is the diagonalization of the fermion kinetic terms. Applying the result of [21], we find that our fermion kinetic Lagrangian can be written as

$$e^{-1}\mathcal{L}_{\text{Fkinetic}} = -\frac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{2}\bar{\chi}\Gamma^\mu D_\mu \chi - \frac{1}{2}(\delta^{IJ} + 4N^{\alpha\beta} A^I_\alpha A^I_\beta)\bar{\lambda}^I \Gamma^\mu \mathcal{D}_\mu \lambda^I \tag{3.51}$$

where the three dimensional fields are given by

$$\begin{aligned}
\psi_a &= e^{-\frac{3g}{2}}(\hat{\psi}_a + \Gamma_a \Gamma^i \hat{\psi}_i) \\
\psi_i &= e^{-\frac{3g}{2}}\big(\hat{\psi}_i - \frac{1}{2}\Gamma_i \hat{\chi}\big) = 2e^{-\frac{3g}{2}+\theta} A^I_i \hat{\lambda}^I, \;\; A^I_i = A^I_\alpha e^{-g}(L^{-1})_i^{\alpha} \\
\chi &= e^{-\frac{3g}{2}}\big(\Gamma^i \hat{\psi}_i + \frac{1}{2}\hat{\chi}\big) \\
\lambda^I &= e^{\frac{\theta}{2}-\frac{3g}{2}}\hat{\lambda}^I \, .
\end{aligned} \tag{3.52}$$

4. Chern-Simons and Yang-Mills gaugings in three dimensions

In this section, we show the on-shell equivalence between non-semisimple Chern-Simons and semisimple Yang-Mills gaugings in three dimensions [5]. We will construct Chern-Simons gauged supergravity with gauge groups $(SO(3) \ltimes \mathbf{R}^3) \times (G \ltimes \mathbf{R}^n)$, $n = \text{dim}G$, and show that the gauging is consistent according to the criterion given in [4]. We then show that this theory is on-shell equivalent to the $SU(2) \times G$ gauged supergravity obtained from $SU(2)$ reduction in the previous section.

Before going to the discussion in details, we give some necessary equations, we will use throughout this section, from [5]. After integrate out the scalars and gauge fields corresponding to the translations or shift symmetries, the non-semisimple Chern-Simons gauged theory with scalar manifold G/H becomes semisimple Yang-Mills gauged supergravity with smaller scalar manifold G'/H'. The resulting Lagrangian is given by [5]

$$\begin{aligned}
e^{-1}\tilde{\mathcal{L}} = &\frac{1}{4}R + e^{-1}h_1 \tilde{\mathcal{L}}_{\text{CS}} - \frac{1}{8}\mathbf{M}_{mn} F^{m\mu\nu} F^n_{\mu\nu} - \frac{1}{4}G_{AB}\tilde{\mathcal{P}}^A_\mu \tilde{\mathcal{P}}^{B\mu} \\
&+ \frac{1}{4}e^{-1}\epsilon^{\mu\nu\rho}\mathbf{M}_{mn}\tilde{\mathcal{V}}^{\underline{n}}{}_A F^m_{\mu\nu}\tilde{\mathcal{P}}^A_\rho - V
\end{aligned} \tag{4.1}$$

where all the notations are the same as that in [5] apart from the metric signature, $(-++)$. We also use our notation for the gauge couplings. We repeat here the

quantities appearing in (4.1)

$$\begin{aligned}
G_{AB} &= \delta_{AB} - \tilde{\mathcal{V}}^{m}_{\ A}\mathbf{M}_{mn}\tilde{\mathcal{V}}^{n}_{\ B}, \qquad \mathbf{M}_{mn} = (\tilde{\mathcal{V}}^{m}_{\ A}\tilde{\mathcal{V}}^{n}_{\ A})^{-1}, \\
\tilde{\mathcal{L}}_{\rm CS} &= \frac{1}{4}\epsilon^{\mu\nu\rho}A^{m}_{\mu}\eta_{mn}\big(\partial_\nu A^n_\rho + \frac{1}{3}g_1 f^{n}_{\ kl}A^k_\nu A^l_\rho\big) \\
&\quad + \frac{1}{4}\epsilon^{\mu\nu\rho}A^{m}_{\mu}\eta_{mn}\big(\partial_\nu A^n_\rho + \frac{1}{3}g_2 f^{n}_{\ kl}A^k_\nu A^l_\rho\big), \\
\tilde{\mathcal{V}}^{\mathcal{M}}_{\ \mathcal{A}}t^{\mathcal{A}} &= \tilde{L}^{-1}t^{\mathcal{M}}\tilde{L}, \\
\tilde{\mathcal{Q}}_\mu + \tilde{\mathcal{P}}_\mu &= \tilde{L}^{-1}(\partial_\mu + g_1\eta_{1mn}A^m_{1\mu}t^n_1 + g_2\eta_{2mn}A^m_{2\mu}t^n_2)\tilde{L}\,.
\end{aligned} \tag{4.2}$$

We are now in a position to construct a consistent Chern-Simons gauged supergravity with gauge groups $(SO(3)\ltimes \mathbf{R}^3)\times(G\ltimes \mathbf{R}^n)$. We proceed as in [10] using the formulation of [4].

The $4(1{+}n)$ scalar fields are described by a coset space $\frac{SO(4,1+n)}{SO(4)\times SO(n+1)}$. We parametrize the coset by

$$L = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \tag{4.3}$$

where A is a symmetric 4×4 matrix, B is a $4\times(n+1)$ matrix, and C is a symmetric $(n+1)\times(n+1)$ matrix. These matrices satisfy the relations

$$\begin{aligned}
&A^2 - BB^t = \mathbb{I}_4, \\
&AB - BC = 0, \\
&C^2 - B^t B = \mathbb{I}_{n+1}\,.
\end{aligned} \tag{4.4}$$

The gauging is characterized by the embedding tensor

$$\Theta_{\mathcal{M}\mathcal{N}} = g_1\delta_{a_1b_1} + g_2\delta_{a_2b_2} + h_1\delta_{b_1b_1} + h_2\delta_{b_2b_2}. \tag{4.5}$$

The ranges of the indices are $a_1, b_1 = 1,2,3$ and $a_2, b_2 = 1,\ldots,n$. We denote the $(5+n)\times(5+n)$ matrix in the block form

$$\left(\begin{array}{c|c} 4\times 4 & 4\times(n+1) \\ \hline (n+1)\times 4 & (n+1)\times(n+1) \end{array}\right). \tag{4.6}$$

With this form, the generators of $SO(4,1+n)$ can be shown as

$$\left(\begin{array}{c|c} J_{SO(4)} & Y \\ \hline Y^t & J_{SO(n+1)} \end{array}\right) \tag{4.7}$$

with Y being non-compact and given by $e_{a\hat{I}} + e_{\hat{I}a}$. We further divide each block by separating its last row and last column from the rest and use the following ranges of indices:

$$\alpha,\, \beta = 1, 2, 3, \qquad I,\, J = 1, \dots, n, \qquad \hat{I} = 5, \dots, n+5, \text{ and } a,\, b = 1, \dots, 4.$$

Various gauge groups are described by the following generators:

$$\begin{aligned} SO(3) &: J^{\alpha}_{a_1} = \epsilon_{\alpha\beta\gamma} e_{\beta\gamma}, \\ G &: J^{I}_{a_2} = f^{I}{}_{JK} e_{JK}, \\ \mathbf{R}^3 &: J^{\alpha}_{b_1} = e_{\alpha,n+5} + e_{n+5,\alpha} + e_{4\alpha} - e_{\alpha 4}, \\ \mathbf{R}^n &: J^{I}_{b_2} = e_{4,I+4} + e_{I+4,4} + e_{n+5,I+4} - e_{I+4,n+5} \\ & \quad \text{with } (e_{ab})_{cd} = \delta_{ac}\delta_{bd}, \text{ etc}\,. \end{aligned} \tag{4.8}$$

Schematically, these gauge generators are embedded in the $(5+n)\times(5+n)$ matrix as

$$\left(\begin{array}{c|c|c|c} J_{a_1}(3\times 3) & \begin{matrix} -b_1 \\ (3\times 1) \end{matrix} & & \begin{matrix} b_1 \\ (3\times 1) \end{matrix} \\ \hline b_1^t(1\times 3) & & b_2^t(1\times n) & \\ \hline & \begin{matrix} b_2 \\ (n\times 1) \end{matrix} & \begin{matrix} J_{a_2} \\ (n\times n) \end{matrix} & \begin{matrix} -b_2 \\ (n\times 1) \end{matrix} \\ \hline b_1^t(1\times 3) & & b_2^t(1\times n) & \end{array}\right) \tag{4.9}$$

where each b_1 and b_2 correspond to various e's factors in J_{b_1} and J_{b_2} in (4.8). Notice that the shift generators have components in both $SO(4)\times SO(n+1)$ and Y parts. Furthermore, J_{b_1} and J_{b_2} transform as adjoint representations of the gauge groups $SO(3)$ and G, respectively.

From this information, we can construct T-tensors and check the consistency of the gauging according to the criterion given in [4], $\mathbb{P}_{\boxplus} T^{IJ,KL} = 0$. The consistency requires that

$$h_2 = -h_1\,. \tag{4.10}$$

The $4(1+n)$ scalars correspond to the non-compact generators Y. After using the shift symmetries to remove some of the shifted scalars and gauge fields, we are left with

$1+3n$ scalars embedded in $(5+n)\times(5+n)$ matrix as

$$\tilde{L} = \left(\begin{array}{c|c|c|c} \frac{1}{\sqrt{\mathbb{I}_3-\mathbf{A}^t\mathbf{A}}} & & \frac{1}{\sqrt{\mathbb{I}_3-\mathbf{A}^t\mathbf{A}}}\mathbf{A}^t & \\ \hline & \cosh\sqrt{2}\Phi & & \sinh\sqrt{2}\Phi \\ \hline \frac{1}{\sqrt{\mathbb{I}_n-\mathbf{A}\mathbf{A}^t}}\mathbf{A} & & \frac{1}{\sqrt{\mathbb{I}_n-\mathbf{A}\mathbf{A}^t}} & \\ \hline & \sinh\sqrt{2}\Phi & & \cosh\sqrt{2}\Phi \end{array}\right). \tag{4.11}$$

Note that in (4.11), we have chosen a specific form of A, B and C. $\mathbf{A}$ is an $n\times 3$ matrix to be identified with A^I_α in the previous section. The resulting coset space is readily recognized as $\mathbf{R}\times\frac{SO(3,n)}{SO(3)\times SO(n)}$ in which Φ corresponds to the $\mathbf{R}\sim SO(1,1)$ part.

In this $(SO(3)\ltimes\mathbf{R}^3)\times(G\ltimes\mathbf{R}^n)$ gauged theory, we find that

$$\tilde{\mathcal{V}}^{\underline{n}}{}_A\tilde{\mathcal{P}}^A_\mu = 0 \tag{4.12}$$

by using $\mathcal{V}$'s given in appendix B and computing $\tilde{\mathcal{P}}^A$ from (4.2). So, there is no coupling term between scalars and gauge field strength in (4.1). Another consequence of this is that the scalar metric G_{AB} in (4.1) is effectively δ_{AB}.

From (B.1), we can compute the scalar manifold metric which is given by the general expression

$$ds^2 = \frac{1}{8}\mathrm{Tr}(\tilde{L}^{-1}d\tilde{L}|_Y\tilde{L}^{-1}d\tilde{L}|_Y) \tag{4.13}$$

where $|_Y$ means that we take the coset component of the corresponding one-form. Using the relation $\mathbf{A}^t\frac{1}{\sqrt{\mathbb{I}_n-\mathbf{A}\mathbf{A}^t}} = \frac{1}{\sqrt{\mathbb{I}_3-\mathbf{A}^t\mathbf{A}}}\mathbf{A}^t$, we find, after a straightforward calculation,

$$\tilde{L}^{-1}d\tilde{L}|_Y = \left(\begin{array}{c|c|c|c} 0 & & \frac{1}{\sqrt{\mathbb{I}_3-\mathbf{A}^t\mathbf{A}}}d\mathbf{A}^t\frac{1}{\sqrt{\mathbb{I}_n-\mathbf{A}\mathbf{A}^t}} & \\ \hline & 0 & & \sqrt{2}d\Phi \\ \hline \frac{1}{\sqrt{\mathbb{I}_n-\mathbf{A}\mathbf{A}^t}}d\mathbf{A}\frac{1}{\sqrt{\mathbb{I}_3-\mathbf{A}^t\mathbf{A}}} & & 0 & \\ \hline & \sqrt{2}d\Phi & & 0 \end{array}\right) \tag{4.14}$$

where we have given only the coset components to simplify the equation. The scalar metric is then given by

$$ds^2 = \frac{1}{2}d\Phi d\Phi + \frac{1}{4}\mathrm{Tr}\bigg(\frac{1}{\mathbb{I}_3-\mathbf{A}^t\mathbf{A}}d\mathbf{A}^t\frac{1}{\mathbb{I}_n-\mathbf{A}\mathbf{A}^t}d\mathbf{A}\bigg). \tag{4.15}$$

This is exactly the same scalar metric appearing in the scalar kinetic terms in (3.49). The scalar matrix appearing in the gauge field kinetic terms can be computed as follows. From (4.1) and (4.2), we can write

$$\mathbf{M}_{mn} = \bar{\mathbf{M}}^{-1}_{mn} \qquad \text{where} \qquad \bar{\mathbf{M}}_{mn} = \tilde{\mathcal{V}}^{\underline{m}}{}_{A}\tilde{\mathcal{V}}^{\underline{n}}{}_{A}\,. \tag{4.16}$$

In our case, the indices $\underline{m},\underline{n} = b_1, b_2$, and $m, n = a_1, a_2$. With properly normalized coset generators Y^A, we find that

$$\bar{\mathbf{M}} = \begin{pmatrix} \bar{\mathbf{M}}_{a_1a_1} & \bar{\mathbf{M}}_{a_1a_2} \\ \bar{\mathbf{M}}_{a_2a_1} & \bar{\mathbf{M}}_{a_2a_2} \end{pmatrix}, \qquad \bar{\mathbf{M}}_{a_{\hat{i}}a_{\hat{j}}} = \tilde{\mathcal{V}}^{b_{\hat{i}}}{}_{A}\tilde{\mathcal{V}}^{b_{\hat{j}}}{}_{A}$$
$$\text{where } \hat{i},\ \hat{j} = 1,2 \text{ and } \tilde{\mathcal{V}}^{b_{\hat{i}}}{}_{A} = \mathrm{Tr}(\tilde{L}^{-1}J_{b_{\hat{i}}}\tilde{L}Y^A). \tag{4.17}$$

After some algebra, we find that the matrix $\mathbf{M}_{a_{\hat{i}}a_{\hat{j}}}$ is the same as $\mathcal{M}_{\mathcal{AB}}$ in (3.50). So, the reduced scalar coset from the Chern-Simons gauged theory is the same as that in the Yang-Mills gauged theory obtained form the $SU(2)$ reduction.

Finally, we have to check the scalar potential. From the embedding tensor, we can compute the potential by using [4]

$$V = A_1^{\bar{I}\bar{J}}A_1^{\bar{I}\bar{J}} - 2g^{ij}A_{2i}^{\bar{I}\bar{J}}A_{2j}^{\bar{I}\bar{J}}. \tag{4.18}$$

We find that potential obtained here is exactly the same as in (3.42). We give the details of this calculation in appendix B. We have now completely shown that the Chern-Simons gauged theory constructed in this section is the same as the Yang-Mills gauged theory obtained from the $SU(2)$ reduction in the previous section.

5. Conclusions

We have obtained Yang-Mills $SU(2)\times G$ gauged supergravity in three dimensions from $SU(2)$ group manifold reduction of six dimensional (1,0) supergravity coupled to an anti-symmetric tensor and G Yang-Mills multiplets. We have also given consistent truncations in both bosonic and fermionic fields from which the resulting consistent reduction ansatz followed. The truncation, which removes three dimensional massive vector fields, results in an $N = 4$ supergravity theory describing $4(1 + \dim G)$ bosonic propagating degrees of freedom, $1 + 3\dim G$ scalars and $3 + \dim G$ gauge fields, together with $4(1 + \dim G)$ fermions. The scalar fields are coordinates in the coset space $\mathbf{R}\times\frac{SO(3,\dim G)}{SO(3)\times SO(\dim G)}$.

Furthermore, we have explicitly constructed the $N = 4$ Chern-Simons $(SO(3)\ltimes \mathbf{R}^3)\times(G\ltimes\mathbf{R}^{\dim G})$ gauged supergravity in three dimensions, following the general procedure detailed in [4]. The scalar manifold $\frac{SO(4,1+\dim G)}{SO(4)\times SO(1+\dim G)}$ becomes $\mathbf{R}\times\frac{SO(3,\dim G)}{SO(3)\times SO(\dim G)}$

after removing the scalars corresponding to the translations or shift symmetries. We have shown the agreement between the resulting Lagrangian and the Lagrangian obtained from dimensional reduction i.e. the gauge field kinetic terms, the scalar manifold metrics and scalar potentials.

We have not given the supersymmetry transformations of the three dimensional fields here. These can, in principle, be obtained by direct computations or using the results in [21] with our truncations. Although supersymmetry transformations of fermions are essential, for example for finding BPS solutions, it is more convenient to work with the equivalent Chern-Simons gauged theory as the latter turns out to be simpler than the equivalent Yang-Mills theory, see [5] for a discussion. In particular, the consistency of the Chern-Simons gauging is encoded in a single algebraic condition on the embedding tensor [1, 2, 3, 4].

In the case where $G = SU(2)$, the $SU(2) \times SU(2)$ Yang-Mills gauged theory is the same as $(SU(2) \ltimes \mathbf{R}^3)^2$ Chern-Simons gauged theory with scalar manifold $\frac{SO(4,4)}{SO(4)\times SO(4)}$. Such quaternionic space has been considered in [10], however the $(SU(2) \ltimes \mathbf{R}^3)^2$ gauging appearing there is different from the one in this paper. The two gauged $SU(2)$'s in [10] are the diagonal subgroups of the two $SU(2)_L$ and the two $SU(2)_R$ respectively of the $SO(4) \times SO(4)$. The latter can be constructed using the parametrization of the target space in terms of e and B matrices as in [10]. The action of shift symmetry generators is to shift B. We can simply set $B = 0$ in this parametrization to obtain the Yang-Mills coset. Although the identification of (A^I_α, Φ) and e is complicated, with the help of *Mathematica*, it can be shown that the two theories are indeed equivalent.

It is interesting to study RG flow solutions in both Chern-Simons and Yang-Mills gauged theories, which, by the present results, can be lifted up to six dimensions. We will report the results of this analysis in a forthcoming paper [22]. A natural open problem is how to obtain 3D $N = 4$ gauged supergravity with two quaternionic scalar manifolds, for example to recover the theory studied in [10]. Presumably we would need to add hypermultiplets to the six dimensional theory, whose scalars themselves live on a quaternionic manifold, or perhaps, we may even need to start with extended supersymmetry in six dimensions.

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A. Details of the calculations

In this appendix, we present some details of the calculations mentioned in section 3.

A.1 Supersymmetry of the fermionic truncation

In order to check the supersymmetry transformation of (3.16), we start by putting our ansatz to the $\delta\hat{\psi}_i$, $\delta\hat{\chi}$ and $\delta\hat{\lambda}^I$ given in (2.2). The result is

$$\begin{aligned}
\delta\hat{\psi}_i =& \frac{1}{8}g_1 e^{g-2f}\not{F}^i(1\otimes 1\otimes 1)\epsilon - \frac{i}{2}e^{-f}(\not{P}_{ij} - \not{\partial}g\delta_{ij})(1\otimes\gamma^j\otimes\sigma_3)\epsilon \\
&+\frac{i}{2}e^{-g}\big(T_{ij} - \frac{1}{2}T\delta_{ij}\big)(1\otimes\gamma^j\otimes 1)\epsilon + e^{\theta}\big[\frac{1}{8}e^{-2f-g}(L^{-1})_j^\alpha\not{\tilde{F}}^\alpha(1\otimes\gamma^j\otimes\sigma_2) \\
&+\frac{1}{4}\tilde{h}(1\otimes 1\otimes\sigma_1) + \frac{i}{4}e^{-f-2g}(L^{-1})_l^\beta(L^{-1})_j^\gamma\epsilon_{ljk}A_\gamma^I\not{D}A_\beta^I(1\otimes\gamma^k\otimes\sigma_1) \\
&+\frac{i}{4}\tilde{a}e^{-3g}(1\otimes 1\otimes\sigma_2)\big](1\otimes\gamma^i\otimes\sigma_2)\epsilon
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\delta\hat{\chi} =& \frac{1}{2}\not{\partial}\theta\epsilon - e^{\theta}\big[\frac{1}{2}\tilde{h}(1\otimes 1\otimes\sigma_1) + \frac{1}{4}e^{-2f-g}(L^{-1})_i^\alpha\not{\tilde{F}}^\alpha(1\otimes\gamma^i\otimes\sigma_2) \\
&+\frac{i}{2}e^{-f-2g}(L^{-1})_i^\beta(L^{-1})_j^\gamma\epsilon_{ijk}A_\gamma^I\not{D}A_\beta^I(1\otimes\gamma^k\otimes\sigma_1) \\
&+\frac{i}{2}\tilde{a}e^{-3g}(1\otimes 1\otimes\sigma_2)]\epsilon
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\delta\hat{\lambda}^I =& \frac{1}{4}[e^{-2f}\not{\tilde{F}}^I(1\otimes 1\otimes 1) + 2ie^{-f-g}(L^{-1})_i^\alpha\not{D}A_\alpha^I(1\otimes\gamma^i\otimes\sigma_3) \\
&+ie^{-2g}(L^{-1})_i^\alpha(L^{-1})_j^\beta\epsilon_{ijk}\mathcal{F}_{\alpha\beta}^I(1\otimes\gamma^k\otimes 1)]\epsilon
\end{aligned} \tag{A.3}$$

We have used the notations $\hat{\not{F}}^I = \hat{F}^I_{MN}\Gamma^{MN}$ etc. From these equations and $\mathbb{I}_2\otimes\mathbb{I}_2\otimes\sigma_3\epsilon^A = \epsilon^A$, we find, up to leading order in fermions, that

$$\delta\hat{\psi}_i - \frac{1}{2}\Gamma_i\delta\hat{\chi} - 2e^{\theta-g}A_\alpha^I(L^{-1})_i^\alpha\delta\hat{\lambda}^I = 0 \tag{A.4}$$

provided that

$$h_{\alpha\beta} = e^{\theta-2g}(12a\delta_{\alpha\beta} - 2A_\alpha^I A_\beta^I) \equiv e^{\theta-2g}N_{\alpha\beta}. \tag{A.5}$$

In proving this result, the following relations are useful

$$\begin{aligned}
L_\alpha^i L_\beta^j T_{ij} &= e^{2\theta-4g}N_{\alpha\gamma}N_{\beta\gamma} \\
T = T_{ii} &= e^{\theta-2g}N_{\alpha\alpha} \\
L_\alpha^i L_\beta^j P_{aij} &= \frac{1}{2}D_a(e^{\theta-2g}N_{\alpha\beta}).
\end{aligned} \tag{A.6}$$

A.2 Supersymmetry of the bosonic truncation

To check (3.20), we start by noting that

$$\hat{g}_{\alpha\beta} = e^{2g} h_{\alpha\beta} = e^{\theta}(12a\delta_{\alpha\beta} - 2A^I_\alpha A^I_\beta). \tag{A.7}$$

It follows that, with $\hat{g}_{\alpha\beta} = \frac{1}{2}(\Gamma_\alpha\Gamma_\beta + \Gamma_\beta\Gamma_\alpha)$,

$$\begin{aligned}
\delta\hat{g}_{\alpha\beta} &= \delta\theta\hat{g}_{\alpha\beta} - 2e^{\theta}(A^I_\alpha \delta A^I_\beta + \delta A^I_\alpha A^I_\beta), \text{ or} \\
\bar{\epsilon}(\Gamma_\alpha\psi_\beta + \Gamma_\beta\psi_\alpha) &= \delta\theta\hat{g}_{\alpha\beta} - 2e^{\theta}(A^I_\alpha \delta A^I_\beta + A^I_\beta \delta A^I_\alpha) \\
&= \bar{\epsilon}\frac{1}{2}(\Gamma_\alpha\Gamma_\beta + \Gamma_\beta\Gamma_\alpha)\chi + 2e^{\theta}\bar{\epsilon}(\Gamma_\beta\lambda^I A^I_\alpha + \Gamma_\alpha\lambda^I A^I_\beta), \\
\text{or} \qquad & \bar{\epsilon}\Gamma_\alpha\left(\psi_\beta - \frac{1}{2}\Gamma_\beta\chi - 2e^{\theta}A^I_\beta\lambda^I\right) + (\alpha \leftrightarrow \beta) = 0
\end{aligned} \tag{A.8}$$

where we have temporarily dropped the hats on the fermions in order to simplify the equations.

We then move to the supersymmetry transformations of $\hat{b}_{MN}$. It is more convenient to work with the transformation of the filed strength $\hat{G}_3$. With equation (3.16), the component $\delta\hat{b}_{\alpha\beta}$ vanishes identically. The $\delta\hat{G}_{3\mu\alpha\beta}$ gives the condition

$$\delta\hat{b}_{\mu\alpha} = \delta(A^I_\alpha A^I_\mu - 6ag_1 A^\alpha_\mu)\,. \tag{A.9}$$

Using $A^I_\mu = \hat{A}^I_\mu + g_1 A^I_\alpha A^\alpha_\mu$ and $\delta\hat{b}_{\mu\alpha}$ from (2.2), we find that

$$\begin{aligned}
\delta\hat{b}_{\mu\alpha} - \delta(A^I_\alpha A^I_\mu - 6ag_1 A^\alpha_\mu) &= 2A^I_\alpha\bar{\epsilon}\Gamma_\mu\hat{\lambda}^I - e^{-\theta}\bar{\epsilon}\Gamma_\mu\hat{\psi}_\alpha + \frac{1}{2}e^{-\theta}\bar{\epsilon}(\Gamma_{\mu\alpha} + \hat{g}_{\mu\alpha})\hat{\chi} \\
&= -e^{-\theta}\bar{\epsilon}\Gamma_\mu\big(\hat{\psi}_\alpha - \frac{1}{2}\Gamma_\alpha\hat{\chi} - 2e^{\theta}A^I_\alpha\hat{\lambda}^I\big) = 0
\end{aligned} \tag{A.10}$$

where we have used

$$\hat{g}_{\mu\alpha} = \hat{e}^i_\mu \hat{e}^i_\alpha = -g_1 h_{\alpha\beta} e^{2g} A^\beta_\mu = -g_1 e^{\theta} N_{\alpha\beta} A^\beta_\mu\,. \tag{A.11}$$

Note that

$$\hat{\psi}_i - \frac{1}{2}\Gamma_i\hat{\chi} - 2e^{\theta-g}A^I_\alpha (L^{-1})^\alpha_i \hat{\lambda}^I = e^{-g}(L^{-1})^\alpha_i\big(\hat{\psi}_\alpha - \frac{1}{2}\Gamma_\alpha\hat{\chi} - 2e^{\theta}A^I_\alpha\hat{\lambda}^I\big). \tag{A.12}$$

$\delta\hat{G}_{3\mu\nu\alpha}$ is simply the derivative of the previous result namely

$$\delta\hat{G}_{3\mu\nu\alpha} = 2\partial_{[\mu}\delta\hat{b}_{\nu]\alpha} = 2\partial_{[\mu}\delta(A^I_{\nu]}A^I_\alpha - 6ag_1 A^\alpha_{\nu]}). \tag{A.13}$$

B. Essential formulae for $N = 4$, $(SO(3) \ltimes \mathbf{R}^3) \times (G \ltimes \mathbf{R}^n)$ gauged supergravity

In this appendix, we give some details of the calculation and needed quantities in the $N = 4$, $(SO(3) \ltimes \mathbf{R}^3) \times (G \ltimes \mathbf{R}^n)$ gauged supergravity constructed in section 4. We recall here the useful expressions for coset space

$$\begin{aligned} L^{-1}D_\mu L &= \frac{1}{2}Q_\mu^{\bar{I}\bar{J}}X^{\bar{I}\bar{J}} + Q_\mu^\alpha X^\alpha + e_\mu^A Y^A\,, \\ L^{-1}t^{\mathcal{M}}L &= \frac{1}{2}\mathcal{V}^{\mathcal{M}\bar{I}\bar{J}}X^{\bar{I}\bar{J}} + \mathcal{V}^{\mathcal{M}}_{\ \alpha}X^\alpha + \mathcal{V}^{\mathcal{M}}_{\ A}Y^A. \end{aligned} \tag{B.1}$$

The tensors A_1 and A_2 can be obtained from the T-tensors by using, with $N = 4$,

$$\begin{aligned} A_1^{\bar{I}\bar{J}} &= -\frac{4}{N-2}T^{\bar{I}\bar{M},\bar{J}\bar{M}} + \frac{2}{(N-1)(N-2)}\delta^{\bar{I}\bar{J}}T^{\bar{M}\bar{N},\bar{M}\bar{N}}, \\ A_{2j}^{\bar{I}\bar{J}} &= \frac{4}{N(N-2)}f^{\bar{M}(\bar{I}m}_{\quad\ j}T^{\bar{J})\bar{M}}_{\quad\ m} + \frac{2}{N(N-1)(N-2)}\delta^{\bar{I}\bar{J}}f^{\bar{K}\bar{L}\ \ m}_{\quad j}T^{\bar{K}\bar{L}}_{\quad m} + \frac{2}{N}T^{\bar{I}\bar{J}}_{\quad j} \end{aligned} \tag{B.2}$$

where $\bar{I}, \bar{J}, \ldots = 1, \ldots 4$ label the R-symmetry indices. The coordinate index on the target space i will be denoted by a pair of indices specifying the entries of the L. In order to simplify the equations, we introduce a symbolic notation R for the R-symmetry generators including their indices. We start by giving all the $\mathcal{V}^{\mathcal{M}}_{\ \bar{I}\bar{J}}$'s.

$$\begin{aligned} \mathcal{V}^R_{a_\alpha} &= \frac{1}{4}\epsilon_{\alpha\beta\gamma}(ARA)_{\gamma\beta}, \qquad a_\alpha = \epsilon_{\alpha\beta\gamma}e_{\beta\gamma}, \\ \mathcal{V}^R_{a_I} &= -\frac{1}{4}f^I_{\ JK}(B^tRB)_{JK}, \qquad \mathcal{V}^R_{b_\alpha} = \frac{1}{2}\mathcal{H}(AR)_{\alpha 4},\ \ \mathcal{H} = A_{44} - B_{4,n+1}, \\ \mathcal{V}^R_{b_I} &= \frac{1}{2}\mathcal{H}(B^tR)_{I4}\,. \end{aligned} \tag{B.3}$$

The $\mathcal{V}^{\mathcal{M}}_{\ i}$'s are given by

$$\begin{aligned} &\mathcal{V}^{a_\alpha}_{\delta L} = \epsilon_{\alpha\beta\gamma}B_{\gamma L}, \qquad \mathcal{V}^{a_I}_{\delta M} = -f_{IJK}B_{\delta J}C_{KM}, \qquad \mathcal{V}^{b_\alpha}_{\delta,n+1} = \mathcal{H}A_{\delta\alpha}, \\ &\mathcal{V}^{b_\alpha}_{4L} = \mathcal{H}B_{\alpha L}, \qquad \mathcal{V}^{b_I}_{4L} = \mathcal{H}C_{IL}, \qquad \mathcal{V}^{b_I}_{\delta,n+1} = \mathcal{H}B_{\delta I}. \end{aligned} \tag{B.4}$$

The T-tensors are defined by

$$T_{\mathcal{AB}} = \Theta_{\mathcal{MN}}\mathcal{V}^{\mathcal{M}}_{\ \mathcal{A}}\mathcal{V}^{\mathcal{N}}_{\ \mathcal{B}} \tag{B.5}$$

which gives

$$\begin{aligned}
T^{RR'} &= \frac{1}{16}\big[-8g_1 R_{\alpha 4}R'_{\alpha 4}\mathrm{det}A + 8g_2\mathcal{H}\frac{B^3}{6} + 4h_1\mathcal{H}^2 R_{\gamma 4}R'_{\gamma 4}\big],\\
T^R_{\delta,n+1} &= \frac{1}{4}\big[-2g_1\mathcal{H}\mathrm{det}AR_{\delta 4} + 2g_2\mathcal{H}\frac{B^3}{6}R_{\delta 4} + 2h_1\mathcal{H}^2 R_{\delta 4}\big],\\
T^R_{4L} &= \frac{1}{4}\big[-g_1\mathcal{H}\epsilon_{\alpha\beta\gamma}B_{\alpha L}(ARA)_{\beta\gamma} + g_2\mathcal{H}f^I{}_{JK}C_{IL}(B^tRB)_{JK}\big],\\
T^R_{\delta L} &= \frac{1}{4}\big[2g_1\mathcal{H}\epsilon_{\alpha\beta\gamma}A_{\delta\beta}B_{\gamma L}(AR)_{\alpha 4} - 2g_2\mathcal{H}f_{IJK}B_{\delta J}C_{KL}(B^tR)_{I4}\big]
\end{aligned} \tag{B.6}$$

where $B^3 = \epsilon_{\alpha\beta\gamma}f_{IJK}B_{\alpha I}B_{\beta J}B_{\gamma K}$. Before moving on, we note the useful relations

$$\begin{aligned}
R_{\alpha\beta} &= \epsilon_{\alpha\beta\gamma}R_{\gamma 4}, \qquad (R^{\bar{K}(\bar{I}}R^{\bar{J})\bar{K}})_{a4} = 3\delta^{\bar{I}\bar{J}}\delta_{a4},\\
R^{\bar{K}(\bar{I}}_{i4}R^{\bar{J})\bar{K}}_{j4} &= -\delta_{i\alpha}\delta_{j\alpha}\delta^{\bar{I}\bar{J}}, \qquad R^{\bar{K}(\bar{I}}_{[i|l|}R^{\bar{J})\bar{K}}_{j]4} = \delta_{\alpha l}\epsilon_{\alpha ij}\delta^{\bar{I}\bar{J}},\\
R^{\bar{K}(\bar{I}}_{\alpha 4}R^{\bar{J})\bar{K}}_{\beta 4} &= -\delta_{\alpha\beta}\delta^{\bar{I}\bar{J}}\,.
\end{aligned} \tag{B.7}$$

The following combination is useful in computing $A^{\bar{I}\bar{J}}_{2i}$

$$\begin{aligned}
f^{\bar{K}(\bar{I}\ \ j}_{\ \ \ 4,n+1}T^{\bar{J})\bar{K}}_j &= \frac{3}{2}\delta^{\bar{I}\bar{J}}\Big(-g_1\mathcal{H}\mathrm{det}A + \frac{1}{6}g_2\mathcal{H}B^3 + h_1\mathcal{H}^2\Big),\\
f^{\bar{K}(\bar{I}\ \ j}_{\ \ \ \delta L}T^{\bar{J})\bar{K}}_j &= -\frac{3}{4}\delta^{\bar{I}\bar{J}}\mathcal{H}(g_1\epsilon_{\alpha\beta\gamma}\epsilon_{\delta\beta'\gamma'}A_{\beta\beta'}A_{\gamma\gamma'}B_{\alpha L}\\
&\quad -g_2 f_{IJK}B_{\beta J}B_{\gamma K}\epsilon_{\delta\beta\gamma}C_{IL}).
\end{aligned} \tag{B.8}$$

We then find A_1 and A_2 tensors

$$\begin{aligned}
A^{\bar{I}\bar{J}}_1 &= -2T\delta^{\bar{I}\bar{J}},\\
A^{\bar{I}\bar{J}}_{2i} &= \frac{1}{2}T^{\bar{I}\bar{J}}_i + \frac{1}{6}X_i\delta^{\bar{I}\bar{J}}
\end{aligned} \tag{B.9}$$

where we have defined the following quantities

$$\begin{aligned}
T &= 2\Big(-g_1\mathcal{H}\mathrm{det}A + \frac{1}{6}g_2\mathcal{H}B^3 + \frac{1}{2}h_1\mathcal{H}^2\Big),\\
X_{4,n+1} &= \frac{3}{2}\Big(-g_1\mathcal{H}\mathrm{det}A + \frac{1}{6}g_2\mathcal{H}B^3 + h_1\mathcal{H}^2\Big),\\
X_{\delta L} &= -\frac{3}{4}\mathcal{H}(g_1\epsilon_{\alpha\beta\gamma}\epsilon_{\delta\beta'\gamma'}A_{\beta\beta'}A_{\gamma\gamma'}B_{\alpha L}\\
&\quad -g_2 f_{IJK}B_{\beta J}B_{\gamma K}\epsilon_{\delta\beta\gamma}C_{IL}).
\end{aligned} \tag{B.10}$$

Using (4.18), we can compute the potential

$$V = 16T^2 - 2\Big(\frac{1}{4}T^{\bar{I}\bar{J}}_i T^{\bar{I}\bar{J}}_i + \frac{1}{9}X_iX_i\Big). \tag{B.11}$$

After some manipulations, we can show that the resulting potential is the same as (3.43) with the following identifications

$$\mathcal{H} \to N^{\frac{1}{6}} e^{-4g} = e^{-\sqrt{2}\Phi}, \qquad A \to \left(\begin{array}{c|c} \frac{1}{\sqrt{\mathbb{I}_3 - \mathbf{A}^t\mathbf{A}}} & 0 \\ \hline 0 & \cosh\sqrt{2}\Phi \end{array}\right),$$

$$B \to \left(\begin{array}{c|c} \frac{1}{\sqrt{\mathbb{I}_3 - \mathbf{A}^t\mathbf{A}}}\mathbf{A}^t & 0 \\ \hline 0 & \sinh\sqrt{2}\Phi \end{array}\right), \qquad C \to \left(\begin{array}{c|c} \frac{1}{\sqrt{\mathbb{I}_n - \mathbf{A}\mathbf{A}^t}} & 0 \\ \hline 0 & \cosh\sqrt{2}\Phi \end{array}\right). \tag{B.12}$$

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